MTH 301 Final Exam Solutions

1. For each of the following finite groups, describe the Class Equation in the form
$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)],$$
where the g_i represent the distinct conjugacy classes of G not contained in $Z(G)$.

   (a) $G = D_8$

   (b) $G = Q_8$

   Solution. (a) First, we determine the distinct conjugacy classes of
$$D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}.$$
Since $Z(D_8) = \{1, r^2\}$, we have $\mathscr{C}_1 = \{1\}$ and $\mathscr{C}_{r^2} = \{r^2\}$. Moreover, as $srs^{-1} = r^3$, $rsr^{-1} = sr^2$, and $s(sr)s^{-1} = sr^3$, we have $\mathscr{C}_r = \mathscr{C}_{r^3}$, $\mathscr{C}_s = \mathscr{C}_{sr^2}$, and $\mathscr{C}_{sr} = \mathscr{C}_{sr^3}$. Furthermore, since $|\langle r \rangle| = 4$ and $\langle r \rangle \leq C_G(r) < G$, we have $|C_G(r)| = 4$, which would imply that $|\mathscr{C}_r| = |G|/|C_G(r)| = 2$ (see 4.3.3 (iv) of the Lesson Plan). Thus, we have $\mathscr{C}_r = \mathscr{C}_{r^3} = \{r, r^3\}$. By a similar reasoning, we can infer that $|C_G(s)| = |C_G(sr)| = 4$, and so it follows that $\mathscr{C}_s = \mathscr{C}_{sr^2} = \{s, sr^2\}$, and $\mathscr{C}_{sr} = \mathscr{C}_{sr^3} = \{s, sr^2\}$. Therefore, the distinct conjugacy classes of D_8 are:
$$\{\{1\}, \{r^2\}, \{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}\}.$$
Finally, the Class Equation in D_8 takes the form
$$8 = |\{1, r^2\}| + \frac{8}{|C_G(r)|} + \frac{8}{|C_G(s)|} + \frac{8}{|C_G(sr)|} = 2 + 2 + 2 + 2.$$

   (b) We begin by determining the conjugacy classes of
$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$
We know that $Z(Q_8) = \{\pm 1\}$ and $\mathscr{C}_1 = \{1\}$. Moreover, for any $x \in \{\pm i, \pm j, \pm k\}$, since $|\langle x \rangle| = 4$ and $\langle x \rangle \leq C_G(x) < G$, we have $|\mathscr{C}_x| = 2$. Thus, the distinct conjugacy classes of Q_8 are:
$$\{\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}\}.$$

Therefore, the Class Equation in Q_8 takes the form

$$8 = |\{\pm 1\}| + \frac{8}{|C_G(i)|} + \frac{8}{|C_G(j)|} + \frac{8}{|C_G(k)|} = 2 + 2 + 2 + 2.$$

2. Besides D_{16}, describe two other non-abelian groups of order 16 that arise as the semi-direct product of two groups.

   Solution. Since $\gcd(4, \phi(4)) > 2$, there exists exists a non-trivial homomorphism $\psi : \mathbb{Z}_4 \to \text{Aut}(\mathbb{Z}_4) \cong \mathbb{Z}_2 = \{[1], [-1]\}$, which is determined by $\psi([1]) = [-1]$. Thus, there exists a non-trivial semi-direct product $\mathbb{Z}_4 \ltimes_{-1} \mathbb{Z}_4$ of order 16. To see that this group is not isomorphic to D_{16}, we first note that $\mathbb{Z}_4 \ltimes_{-1} \mathbb{Z}_4$ has a normal cyclic subgroup of order 4 whose quotient is also isomorphic to $\mathbb{Z}_4$. However, the only cyclic subgroup of D_{16} is $\{1, r^2, r^4, r^6\}$, whose quotient isomorphic to the Klein 4-group (Verify!).

   Since $\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \text{GL}(2, \mathbb{Z}_2) \cong S_3$ has no element of order 4, a non-trivial homomorphism $\psi : \mathbb{Z}_4 \to \text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ must satisfy $|\psi([1])| = 2$. Since S_3 has three elements of order 2, there are three possibilities of such a homomorphism. Thus, there exists three non-trivial semi-direct products of the form $\mathbb{Z}_4 \ltimes_\psi (\mathbb{Z}_2 \times \mathbb{Z}_2)$ of order 16 (which are isomorphic, since the elements of order 2 in S_3 are conjugate). In particular, we may choose the semi-direct determined by $\psi([1]) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Furthermore, $\mathbb{Z}_4 \ltimes_\psi (\mathbb{Z}_2 \times \mathbb{Z}_2)$ has a normal subgroup of order 4 that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. As D_{16} as no such subgroups (Verify!), $\mathbb{Z}_4 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) \ncong D_{16}$.

3. Let p and q be distinct primes such that $p > q$ and $q \nmid p^2 - 1$. Then show that a group of order $p^2 q$ is abelian.

   Solution. Let G be a group with $|G| = p^2 q$. Then by the Sylow's theorems, G has a Sylow p-subgroup P of order p^2 and a Sylow q-subgroup Q of order q. Consequently, both P and Q are abelian. Moreover since $n_p \equiv 1 \pmod{p}$ and $n_p \mid q$, we have that $n_p = 1$, which implies that $P \lhd G$. Furthermore, as $n_q \equiv 1 \pmod{q}$, $n_q \mid p^2$, and $q \nmid p^2 - 1$, it follows that $n_q = 1$, and hence $Q \lhd G$. Finally, since $P \cap Q = \{1\}$, we have $G = PQ$. Finally, by Midterm Question 1, it follows that $G = PQ \cong P \times Q$, which shows that G is abelian.

4. Show that a group of order 56 is non-simple.

Solution. Since $56 = 2^3 \cdot 7$, by the Sylow's theorems, a group G of order 56 has a Sylow 2-subgroup H of order 8 and a Sylow 7-subgroup K of order 7. Furthermore, we have $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 8$, which shows that $n_7 \in \{1, 8\}$. If $n_7 = 1$, then clearly $K \lhd G$, which shows that G is non-simple.

Suppose that $n_7 = 8$. Then G has eight distinct subgroups of order 7 (one of which is K) that intersect pairwise trivially. Consequently, G must have 48 distinct elements of order 7. Since as $|G| = 56$, this shows that H is a unique Sylow 2-subgroup of G, and so we have $H \lhd G$. Therefore, G is non-simple.

5. (a) Show that there exists a subgroup of S_8 isomorphic to Q_8.

   (b) For $n \leq 7$, show that there exists no subgroup of S_n that is isomorphic to Q_8.

Solution. (a) We know from Lesson Plan 4.3.2 (i) that the action $G \curvearrowright G$ by left-multiplication is faithful, and so it affords a left-regular representation : $\psi : G \to S(G)$, which is injective. In our case, since $G = Q_8$, we have $S(G) \cong S_{|Q_8|} = S_8$ and $Q_8 \cong \psi(Q_8) < S_8$, from which the assertion follows.

(b) For $n \leq 7$, let us assume that S_n does contain a subgroup $H \cong Q_8$. Since S_n is the group of permutations of the set $\{1, 2, \ldots, n\}$, there is a natural action $S_n \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} : (\sigma, i) \mapsto \sigma(i)$ which affords a permutation representation $S_n \to S(\{1, 2, \ldots, n\})$ that is an isomorphism (see Assignment II: Practice Assignment 1(b)). Since $H < S_n$, this action restricts to faithful action of $H \curvearrowright \{1, 2, \ldots, n\}$. Applying the Orbit-Stabilizer theorem to this action, we get $[H : H_i] = |\mathcal{O}_i|$, for each $i \in \{1, 2, \ldots, n\}$. Since $|\mathcal{O}_i| \leq n < 8$, it follows that H_i is non-trivial. Furthermore, we have $\ker(H \curvearrowright \{1, 2, \ldots, n\}) = \cap_{i \in \{1,2,\ldots,n\}} H_i$. Since the intersection of all non-trivial subgroups of Q_8 is $\{\pm 1\}$, which is non-trivial, it follows that $\ker(H \curvearrowright \{1, 2, \ldots, n\})$ is non-trivial. Consequently, $H \curvearrowright \{1, 2, \ldots, n\}$ is not faithful, which is a contradiction.